TRANSFERRING L^p EIGENFUNCTION BOUNDS FROM S^{2n+1} TO h^n

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Abstract. By using the notion of contraction of Lie groups, we transfer $L^p - L^2$ estimates for joint spectral projectors from the unit complex sphere S^{2n+1} in $\mathbb{C}^{n+1}$ to the reduced Heisenberg group h^n. In particular, we deduce some estimates recently obtained by H. Koch and F. Ricci on h^n. As a consequence, we prove, in the spirit of Sogge's work, a discrete restriction theorem for the sub-Laplacian L on h^n.

1. Introduction

In the last twenty-five years the notion of *contraction* (or *continuous deformation*) of Lie algebras and Lie groups, introduced in 1953 in a physical context by E. Inönu and E. P. Wigner, was developed in a mathematical framework as well. The basic idea is that, given a Lie algebra $\mathfrak{g}_1$, from a family of non-degenerate transformations of its structure constants it is possible to obtain, in a limit sense, a non-isomorphic Lie algebra $\mathfrak{g}_2$.

It turns out that the deformed algebra $\mathfrak{g}_2$ inherits analytic and geometric properties from $\mathfrak{g}_1$ and that the same holds for the corresponding Lie groups. As a consequence, transference results have attracted considerable attention, in particular in the context of Fourier multipliers. In fact, contraction has been successfully used to transfer L^p multiplier theorems from one Lie group to another one. There is an extensive literature on such topic, centered about deLeeuw's theorems; we only mention here the results by A. H. Dooley, G. Gaudry, J. W. Rice and R. L. Rubin ([D], [DGa], [DRi1], [DRi2], [Ru]), concerning, in particular, contraction of rotation groups and semisimple Lie groups.

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The primary purpose of this paper is to show that contraction is an effective tool to transfer L^p eigenfunction bounds as well. In particular, we shall focus on a contraction from the complex unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ to the reduced Heisenberg group h^n.

We recall that, if P is a second order self-adjoint elliptic differential operator on a compact manifold M and if P_λ denotes the spectral projection corresponding to the eigenvalue λ^2, a classical problem is to estimate the norm ν_p of P_λ as an operator from $L^p(M)$, $1 \le p \le 2$, to $L^2(M)$. Sharp estimates for ν_p have been obtained by C. Sogge ([So2]), who proved that

$$||P_\lambda||_{(p,2)} \le C\lambda^{\gamma(\frac{1}{p},n)} \quad 1 \le p \le 2\,, \tag{1.1}$$

where γ is the piecewise affine function on $[\frac{1}{2}, 1]$ defined by

$$\gamma(\frac{1}{p}, n) := \begin{cases} n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} & \text{if } 1 \le p \le \tilde{p} \\ \frac{n-1}{2}(\frac{1}{p} - \frac{1}{2}) & \text{if } \tilde{p} \le p \le 2, \end{cases}$$

with *critical point* $\tilde{p}$ given by $\tilde{p} := 2\frac{n+1}{n+3}$.

The starting point for our approach is a sharp two-parameter estimate for joint spectral projections on complex spheres, recently obtained by the first author ([Ca]). More precisely, we consider the Laplace-Beltrami operator $\Delta_{S^{2n+1}}$ and the Kohn Laplacian $\mathcal{L}$ on S^{2n+1} (this set yields a basis for the algebra of $U(n+1)$-invariant differential operators on S^{2n+1}). It is possible to work out a joint spectral theory. In particular, we denote by $\mathcal{H}^{\ell,\ell'}$, $\ell, \ell' \ge 0$, the joint eigenspace with eigenvalue $\mu_{\ell,\ell'}$ for $\Delta_{S^{2n-1}}$, where $\mu_{\ell,\ell'} := -(\ell+\ell')\,(\ell+\ell'+2n-2)$, and with eigenvalue $\lambda_{\ell,\ell'}$ for $\mathcal{L}$, where $\lambda_{\ell,\ell'} := -2\ell\ell' - (n-1)(\ell+\ell')$ ([Kl]). It is a classical fact ([VK, Ch.11]) that

$$L^2\left(S^{2n+1}\right) = \sum_{\ell,\ell'=0}^{+\infty} \oplus \mathcal{H}^{\ell\ell'}\,, \tag{1.2}$$

where the series on the right converges in the L^2-norm.
By the symbol $\pi_{\ell\ell'}$ we denote the joint spectral projector from $L^2(S^{2n-1})$ onto $\mathcal{H}^{\ell\ell'}$. In [Ca] we proved the following two-parameter L^p eigenfunction bounds

$$||\pi_{\ell,\ell'}||_{(p,2)} \lesssim C\ (2q_\ell + n - 1)^{\alpha(\frac{1}{p},n)}\,(1+Q_\ell)^{\beta(\frac{1}{p},n)} \quad \text{for all } \ell, \ell' \ge 0\,, \tag{1.3}$$

where $Q_\ell := \max\{\ell, \ell'\}$, $q_\ell := \min\{\ell, \ell'\}$ and α and β are the piecewise affine functions represented in Figure 1 at the end of Section 2. We remark that the critical exponent is in our case $\frac{2(2n+1)}{2n+3}$ and cannot be directly deduced from Sogge's results. Observe moreover that $2q_\ell + n - 1$ and Q_ℓ are related to the eigenvalues $\lambda_{\ell,\ell'}$ and $\mu_{\ell,\ell'}$, since they grow, respectively, as $\frac{|\lambda_{\ell,\ell'}|}{\ell+\ell'}$ and $|\mu_{\ell,\ell'}|^{\frac{1}{2}}$.

On the other hand, on the reduced Heisenberg group h^n, defined as $h^n := \mathbb{C}^n \times \mathbb{T}$, with product

$$(\mathbf{z}, e^{it})(\mathbf{w}, e^{it'}) := \left(\mathbf{z} + \mathbf{w}, e^{i(t+t'+\Im m\, \mathbf{z}\bar{\mathbf{w}})}\right)\,,$$

with $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, $t, s \in \mathbb{R}$, we consider the sub-Laplacian L and the operator $i^{-1}\partial_t$. The pairs $(2|m|(2k+1), m)$, with $m \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{N}$, give the discrete joint spectrum of these operators. Recently H. Koch and Ricci proved the following $L^p - L^2$ estimate for the orthogonal projector $P_{m,k}$ onto the joint eigenspace

$$||P_{m,k}||_{(L^p(h^n),L^2(h^n))} \lesssim C\,(2k+n)^{\alpha(\frac{1}{p},n)} \cdot |m|^{\beta(\frac{1}{p},n)}\,, \tag{1.4}$$

$1 \le p \le 2$, where α and β are given by (1.3) ([KoR]).

We start showing in Section 2 that $P_{m,k}$ may be obtained as limit in the L^2-norm of a sequence of joint spectral projectors on S^{2n+1}. Then we give an alternative proof of (1.4) by a contraction argument.

A contraction from $SU(2)$ to the one-dimensional Heisenberg group H^1 was studied by F. Ricci and Rubin ([R], [RRu]). In [Ca] the first author used some ideas from [R] to transfer $L^p - L^2$ estimates for norms of harmonic projection operators from the unit sphere S^3 in $\mathbb{C}^2$ to the reduced Heisenberg group h^1. In this paper we discuss the higher-dimensional case.

A contraction from the unit sphere S^{2n+1} to the Heisenberg group H^n for $n > 1$ was analyzed by Dooley and S. K. Gupta; in a first paper they adapted the notion of Lie group contraction to the homogeneous space $U(n+1)/U(n)$ and described the relationship between certain unitary irreducible representations of $U(n+1)$ and H^n ([DG1]), in a second paper they proved a deLeeuw's type theorem on H^n by transferring results from S^{2n+1} ([DG2]). The contraction we use here is essentially that introduced by Dooley and Gupta; anyway, their approach is mainly algebraic, while our interest is adressed to the analytic features of the problem.

As an application of (1.3) we prove in Section 3 a discrete restriction theorem for the sub-Laplacian L on h^n in the spirit of Sogge's work ([So1], see also (1.1)). More precisely, let Q_N be the spectral projection corresponding to the eigenvalue N associated to L on h^n, that is

$$Q_N f := \sum_{(2k+n)|m|=N} P_{m,k} f\,.$$

The study of $L^p - L^2$ mapping properties of Q_N was suggested by D. Müller in his paper about the restriction theorem on the Heisenberg group ([M]). In [Th1] Thangavelu proved that

$$||Q_N||_{(L^p(h^n),L^2(h^n))} \le C\ (N^n d(N))^{\frac{1}{p}-\frac{1}{2}}\ , \qquad 1 \le p \le 2\,, \tag{1.5}$$

where $d(N)$ is the divisor-type function defined by

$$d(N) := \sum_{2k+n|N} \frac{1}{2k+n}\,, \tag{1.6}$$

and the estimate is sharp for $p = 1$. By $a|b$ we mean that a divides b. Other types of restriction theorems on the Heisemberg group were discussed by Thangavelu in [Th2].

By using orthogonality, we add up the estimates in (1.3) and obtain $L^p - L^2$ bounds for the norm of Q_N, which in some cases improve (1.5). The exponent

appearing in (1.5) is an affine function of $\frac{1}{p}$. In our estimate the exponent of $d(N)$ is, like in Sogge's results, a piecewise affine function of $\frac{1}{p}$. In other words, there is a critical point $\tilde{p}$ where the slope of the exponent changes. This critical point is the same that was found on complex spheres ([Ca]).

Our bounds are in general not sharp. The reason is that with our procedure we disregard the interferences between eigenfunctions. We show however that there are arithmetic progressions N_m in $\mathbb{N}$ for which our estimates for $||Q_{N_m}||_{(p,2)}$ are sharp and better than (1.5). Moreover, since the behaviour of $d(N)$ is highly irregular, we inquire about the average size of $||Q_N||_{(p,2)}$. We prove in this case that $L^p - L^2$ estimates do not involve divisor-type functions and that the critical point disappears.

It is a pleasure to thank Professor Fulvio Ricci for his valuable help.

2. Preliminaries

In this section we introduce some notation and recall a few results, that will be used in the following.

2.1. *Some notation.* For $n \geq 1$ let $\mathbb{C}^{n+1}$ denote the n-dimensional complex space endowed with the scalar product $< \mathbf{z}, \mathbf{w} >:= z_1\bar{w}_1 + \ldots + z_{n+1}\bar{w}_{n+1}$, $\mathbf{z}, \mathbf{w} \in \mathbb{C}^{n+1}$, and let S^{2n+1} denote the unit sphere in $\mathbb{C}^{n+1}$, that is

$$S^{2n+1} := \{\mathbf{z} = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \ :< \mathbf{z}, \mathbf{z} >= 1\}\,.$$

The symbol $\mathbf{1}$ will denote the north pole of S^{2n+1}, that is $\mathbf{1} := (0, \ldots, 0, 1)$.

For every $\ell, \ell' \in \mathbb{N}$ the symbol $\mathcal{H}^{\ell\ell'}$ will denote the space of the restrictions to S^{2n+1} of harmonic polynomials $p(\mathbf{z}, \bar{\mathbf{z}}) = p(z_1, \ldots, z_{n+1}, \bar{z}_1, \ldots, \bar{z}_{n+1})$, of homogeneity degree ℓ in $z_1, \ldots, z_{n+1}$ and of homogeneity degree ℓ' in $(\bar{z}_1, \ldots, \bar{z}_{n+1})$, *i.e.* such that

$$p(a\mathbf{z}, b\bar{\mathbf{z}}) = a^{\ell} b^{\ell'} p(\mathbf{z}, \bar{\mathbf{z}})\,, \quad a, b \in \mathbb{R}\,, \quad \mathbf{z} \in \mathbb{C}^n\,.$$

For a detailed description of the spaces $\mathcal{H}^{\ell\ell'}$ see Chapter 11 in [VK]. We only recall here that a polynomial p in $\mathbf{z}, \bar{\mathbf{z}}$ is said to be harmonic if

$$\Delta_{S^{2n+1}} p := \frac{1}{4}\Big(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \ldots + \frac{\partial^2}{\partial z_{n+1} \partial \bar{z}_{n+1}}\Big) p = 0, \tag{2.1}$$

where $\Delta_{S^{2n+1}}$ denotes the Laplace-Beltrami operator.

A zonal function of bidegree (ℓ, ℓ') on S^{2n+1} is a function in $\mathcal{H}^{\ell\ell'}$, which is constant on the orbits of the stabilizer of $\mathbf{1}$ (which is isomorphic to $U(n)$). Given a zonal function f, we may associate to f a map ${}^b f$ on the unit disk by

$$f(\mathbf{z}) = {}^b f(< \mathbf{z}, \mathbf{1} >)\,,\, \mathbf{z} \in S^{2n+1}\,,$$

(by using the notation in Section 11.1.5 of [VK] we have $< \mathbf{z}, \mathbf{1} >= z_n = e^{i\varphi} \cos\theta$, where $\varphi \in [0, 2\pi]$ and $\theta \in [0, \frac{\pi}{2}]$).

By means of ${}^b f$ we may define a convolution between a zonal function f and an arbitrary function g on S^{2n+1}. More precisely, we set

$$\big(f * g\big)(\mathbf{z}) := \int_{S^{2n+1}} {}^b f(< \mathbf{z}, \mathbf{w} >) g(\mathbf{w}) d\sigma(\mathbf{w})\,,$$

where $d\sigma$ is the measure invariant under the action of the unitary group $U(n+1)$ (see (3.4) for an explicit formula). In the following we shall write $f(\theta,\varphi)$ instead of ${}^b f(e^{i\varphi}\cos\theta)$.

Let $L^2(S^{2n+1})$ be the Hilbert space of functions on S^{2n+1} endowed with the inner product $(f,g) := \int_{S^{2n+1}} f(\mathbf{z})\overline{g(\mathbf{z})}d\sigma(\mathbf{z})$.

It is a classical fact ([VK], Ch. 11) that $L^2(S^{2n+1})$ is the direct sum of the pairwise orthogonal and $U(n+1)$-invariant subspaces $\mathcal{H}^{\ell\ell'}$, $\ell,\ell'\geq 0$. In other words, every $f\in L^2(S^{2n+1})$ admits a unique expansion

$$f=\sum_{\ell,\ell'=0}^{+\infty} Y^{\ell\ell'}\,,$$

where $Y^{\ell\ell'}\in\mathcal{H}^{\ell\ell'}$ for every $\ell,\ell'\geq 0$ and the series at the right converges to f in the $L^2(S^{2n+1})$-norm.

The orthogonal projector onto $\mathcal{H}^{\ell\ell'}$

$$\pi_{\ell,\ell'}\;:L^2(S^{2n-1})\ni f\mapsto Y^{\ell\ell'}\in\mathcal{H}^{\ell\ell'} \tag{2.2}$$

may be written as

$$\pi_{\ell,\ell'}f := {}^b\mathbb{Z}_{\ell,\ell'} * f\,,$$

where $\mathbb{Z}_{\ell,\ell'}$ is the zonal function from $\mathcal{H}^{\ell\ell'}$, given by

$$\begin{gathered}{}^b\mathbb{Z}_{\ell,\ell'}(\theta,\varphi) := \frac{d_{\ell,\ell'}}{\omega_{2n+1}}\frac{q_\ell!(n-1)!}{(q_\ell+n-1)!}e^{i(\ell'-\ell)\varphi}(\cos\theta)^{|\ell-\ell'|}P_{q_\ell}^{(n-1,|\ell-\ell'|)}(\cos 2\theta)\\ \ell\,,\ell'\geq 1,\;\varphi\in[0,2\pi]\,,\;\theta\in[0,\frac{\pi}{2}]\,.\end{gathered} \tag{2.3}$$

where $q_\ell=\min\big(\ell,\ell'\big)$, ω_{2n+1} denotes the surface area of S^{2n+1}, $P_{q_\ell}^{(n-1,|\ell-\ell'|)}$ is the Jacobi polynomial and

$$d_{\ell,\ell'} := \mathrm{dim}\mathcal{H}^{\ell,\ell'} = n\cdot\frac{\ell+\ell'+n}{\ell\ell'}\binom{\ell+n-1}{\ell-1}\binom{\ell'+n-1}{\ell'-1}\quad\text{for all }\ell,\ell'\geq 1.$$

Recall finally that $\mathcal{H}^{\ell,0}$ consists of holomorphic polynomials and $\mathcal{H}^{0,\ell}$ consists of polynomials whose complex conjugates are holomorphic. In both cases, the dimension of the space is given by

$$\dim\mathcal{H}^{\ell,0}=\dim\mathcal{H}^{0,\ell}=\binom{\ell+n-1}{\ell}$$

and the zonal function is

$$\mathbb{Z}_{\ell,0}(\theta,\varphi) := \frac{1}{\omega_{2n-1}}\binom{\ell+n-1}{\ell}e^{-i\ell\varphi}(\cos\theta)^\ell,\quad\varphi\in[0,2\pi],\;\theta\in[0,\frac{\pi}{2}].$$

In this paper we shall adopt the convention that C denotes a constant which is not necessarily the same at each occurrence.

2.2. *Some useful results.* In order to transfer L^p bounds from S^{2n+1} to h^n we shall need both a pointwise estimate for the Jacobi polynomials, due to Darboux and Szegö ([Sz, pgs. 169,198]), and a Mehler-Heine-type formula, relating Jacobi and Laguerre polynomials ([Sz], [R]).

Lemma 2.1. *Let $\alpha, \beta > -1$. Fix $0 < c < \pi$. Then*

$$P_\ell^{(\alpha,\beta)}(\cos\theta) = \begin{cases} O\left(\ell^\alpha\right) & \text{if } 0 \le \theta \le \frac{c}{\ell}, \\ \ell^{-\frac{1}{2}} k(\theta) \left(\cos\left(N_\ell \theta + \gamma\right) + \left(\ell \sin\theta\right)^{-1} O(1)\right) & \text{if } \frac{c}{\ell} \le \theta \le \pi - \frac{c}{\ell} \\ O\left(\ell^\beta\right) & \text{if } \pi - \frac{c}{\ell} \le \theta \le \pi, \end{cases}$$

where $k(\theta) := \pi^{\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}$, $N_\ell := \ell + \frac{\alpha+\beta+1}{2}$, $\gamma := -(\alpha + \frac{1}{2})\frac{\pi}{2}$.

Proposition 2.2. [R, pg.224] *Let $n \ge 1$ and let x be a real number. Fix k and j in $\mathbb{N}$, $j \ge k$. Then*

$$\lim_{N \to +\infty} \cos^{N-j-k}\left(\frac{x}{\sqrt{N-j-k}}\right) \cdot P_k^{(j-k,N-j-k)}\left(\cos\frac{2x}{\sqrt{N-j-k}}\right) = L_k^{j-k}\left(x^2\right) \cdot e^{-\frac{1}{2}x^2}. \tag{2.4}$$

Our proof is based on the following two-parameter estimate for the $L^p - L^2$ norm of the complex harmonic projectors $\pi_{\ell,\ell'}$, defined by (2.2).

Theorem 2.3. [Ca] *Let $n \ge 2$ and let ℓ, ℓ' be non-negative integers. Then*

$$||\pi_{\ell,\ell'}||_{(p,2)} \lesssim C \left(\frac{2\ell\ell' + n(\ell+\ell')}{\ell+\ell'}\right)^{\alpha(\frac{1}{p},n)} (\ell+\ell')^{\beta(\frac{1}{p},n)} \text{ if } 1 \le p \le 2, \tag{2.5}$$

where

$$\alpha(\frac{1}{p}, n) := \begin{cases} n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} & \text{if } 1 \le p < \tilde{p} \\ \frac{1}{4} - \frac{1}{2p} & \text{if } \tilde{p} \le p \le 2, \end{cases} \tag{2.6}$$

with $\tilde{p} = 2\frac{2n+1}{2n+3}$, *and*

$$\beta(\frac{1}{p}, n) = n\left(\frac{1}{p} - \frac{1}{2}\right) \text{ for all } 1 \le p \le 2, \tag{2.7}$$

The above estimates are sharp.

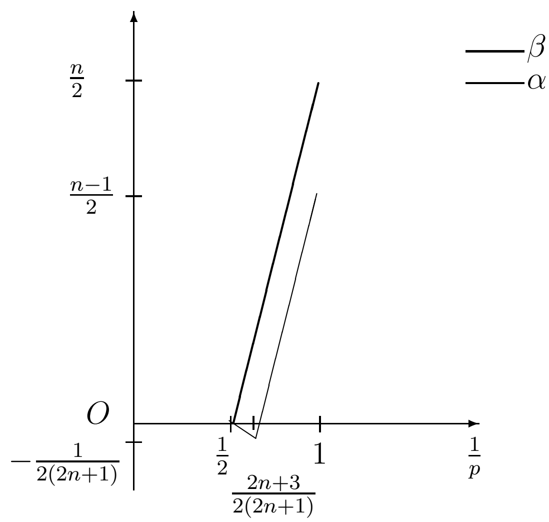

Figure 1. The exponents α and β as functions of $\frac{1}{p}$

3. L^p eigenfunction bounds on H^n

The Heisenberg group H^n is a Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$, endowed with the product

$$(\mathbf{z}, t)(\mathbf{w}, s) := (\mathbf{z} + \mathbf{w}, t + s + \Im m\, \mathbf{z} \cdot \overline{\mathbf{w}}) \ ,$$

with $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, $t, s \in \mathbb{R}$.
We denote an element in H^1 by $(\rho e^{i\varphi}, t)$, where $\rho \in [0, +\infty)$, $\varphi \in [0, 2\pi]$, $t \in \mathbb{R}$, and an element in H^n by $\big(\rho\underline{\eta}, t\big)$, where $\rho \in [0, +\infty)$, $t \in \mathbb{R}$ and $\underline{\eta} \in S^{2n-1}$ is given by

$$\underline{\eta} = \begin{cases} e^{i\varphi_1} \sin\theta_{n-1} \sin\theta_{n-2} \ldots \sin\theta_1 \\ e^{i\varphi_2} \sin\theta_{n-1} \sin\theta_{n-2} \ldots \cos\theta_1 \\ \vdots \\ e^{i\varphi_n} \cos\theta_{n-1}\,, \end{cases} \tag{3.1}$$

with $\varphi_k \in [0, 2\pi]$, $k = 1, \ldots, n$, and $\theta_j \in [0, \frac{\pi}{2}]$, $j = 1, \ldots, n-1$.
Observe that $\underline{\eta} = \underline{\eta}\,(\Theta_{n-1}, \Phi_n)$, where $\Theta_{n-1} := (\theta_1\,, \theta_2\,, \ldots, \theta_{n-1})$ and $\Phi_n := (\varphi_1, \ldots, \varphi_n)$.
Define now a map $\Psi : H^n \to S^{2n+1}$ by

$$\Psi\,:\,\big(\rho\underline{\eta}, t\big) \mapsto (\Theta_{n-1}\,, \rho\,, \Phi_n, t)\,, \tag{3.2}$$

where $(\Theta_{n-1}\,,\rho\,,\Phi_n,t)\in S^{2n+1}$ is given by

$$(\Theta_{n-1}\,,\rho\,,\Phi_n,t):=\begin{cases} e^{i\varphi_1}\sin\rho\,\sin\theta_{n-1}\,\sin\theta_{n-2}\ldots\sin\theta_1\\ e^{i\varphi_2}\sin\rho\,\sin\theta_{n-1}\sin\theta_{n-2}\ldots\cos\theta_1\\ \vdots\\ e^{i\varphi_n}\sin\rho\,\cos\theta_{n-1}\\ e^{it}\cos\rho\,. \end{cases} \tag{3.3}$$

We introduce in this way a coordinate system $(\Theta_{n-1}\,,\rho\,,\Phi_n,t)$ on S^{2n+1}, if ρ and t are restricted, respectively, to $[0,\frac{\pi}{2}]$ and $[-\pi,\pi]$.

The invariant measure $d\sigma_{S^{2n+1}}$ on S^{2n+1} in the spherical coordinates (3.3) is

$$\frac{n!}{2\pi^{n+1}}\Pi_{k=1}^{n}d\varphi_k\;dt\;\sin^{2n-1}\rho\cos\rho\,d\rho\;\Pi_{j=1}^{n-1}\sin^{2j-1}\theta_j\cos\theta_j\,d\theta_j. \tag{3.4}$$

The factor $\frac{n!}{2\pi^{n+1}}$ is introduced in order to make the measure of the whole sphere equal to 1.
The Haar measure on H^n in these coordinates is

$$\frac{n!}{2\pi^{n+1}\sqrt{\omega_{2n+1}}}\rho^{2n-1}d\rho\,d\varphi_1\ldots d\varphi_n\,\Pi_{j=1}^{n-1}\sin^{2j-1}\theta_j\cos\theta_j\,d\theta_j.$$

The *reduced Heisenberg group* h^n is defined as $h^n:=\mathbb{C}^n\times\mathbb{T}$, with product

$$(\mathbf{z},e^{it})(\mathbf{w},e^{it'}):=\left(\mathbf{z}+\mathbf{w},e^{i(t+t'+\Im m\,\mathbf{z}\bar{\mathbf{w}})}\right)\,,$$

with $\mathbf{z},\mathbf{w}\in\mathbb{C}^n$, $t,s\in\mathbb{R}$.
Let now f be a function on h^n, with compact support. Let $\tilde f$ be the function f extended by periodicity on $\mathbb{R}$ with respect to the variable t. Define the function f_ν on S^{2n+1} by

$$f_\nu(\rho\,,\Theta_{n-1}\,,\Phi_n,t):=\nu^n\,\tilde f(\rho\sqrt{\nu}\,\underline{\eta},t\nu\,)\,,\;\;\nu\in\mathbb{N}. \tag{3.5}$$

Lemma 3.1. *Let f be an integrable function on h^n with compact support. If $1\le p\le+\infty$, then*

$$\nu^{-\frac{n}{p'}}||f_\nu||_{L^p(S^{2n+1})}<||f||_{L^p(h^n)}\quad\textit{and}$$

$$\lim_{\nu\to+\infty}\nu^{-\frac{n}{p'}}||f_\nu||_{L^p(S^{2n+1})}=||f||_{L^p(h^n)}\,.$$

Proof. The proof is similar to that of Lemma 2 in [RRu] and is omitted. Compare also with Lemma 4.3 in [DG2]. □

Throughout the paper we shall consider a pair of strongly commuting operators on h^n. The first is the left-invariant sub-Laplacian L, defined by

$$L:=-\sum_{j:=1}^{n}\left(X_j^2+Y_j^2\right)\,,$$

where $X_j:=\partial_{x_j}-y_j\partial_t$ and $Y_j:=\partial_{y_j}+x_j\partial_t$. The second is the operator $T:=i^{-1}\partial_t$. These operators generate the algebra of differential operators on h^n

invariant under left translation and under the action of the unitary group. One can work out a joint spectral theory; the pairs $(2|m|(2k+n), m)$, with $m \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{N}$, give the discrete joint spectrum of L and $i^{-1}\partial_t$. We shall denote by $P_{m,k}$ the orthogonal projector onto the joint eigenspace.

By considering the Fourier decomposition of functions in $L^2(h^n)$ with respect to the central variable, we obtain an orthogonal decomposition of $L^2(h^n)$ as

$$L^2(h^n) = \mathcal{H}_0 \oplus \mathcal{H}\,,$$

where $\mathcal{H}$ is given by

$$\mathcal{H} := \{f \in L^2(h^n) \,:\, \int_{\mathbb{T}} f(z,t)dt = 0\,\}\,.$$

The projectors $P_{m,k}$ map $L^2(h^n)$ onto $\mathcal{H}$ and provide a spectral decomposition for $\mathcal{H}$. We point the attention on this decomposition, since the spectral analysis of L on $\mathcal{H}_0$ essentially reduces to the analysis of the Laplacian on $\mathbb{C}^n$.

On the complex sphere S^{2n+1} the algebra of $U(n+1)$-invariant differential operators is commutative and generated by two elements; a basis is given by the Laplace-Beltrami operator $\Delta_{S^{2n+1}}$, defined by (2.1), and the Kohn Laplacian $\mathcal{L}$ on S^{2n+1}, defined by

$$\mathcal{L} := \sum_{j<k} M_{jk}\overline{M}_{jk} + \overline{M}_{jk}M_{jk}\,,$$

with

$$M_{jk} := \overline{z}_j\partial_{z_k} - \overline{z}_k\partial_{z_j} \quad \text{and} \quad \overline{M}_{jk} := z_j\partial_{\overline{z}_k} - z_k\partial_{\overline{z}_j}\,.$$

We shall call $\mathcal{H}^{\ell,\ell'}$ the joint eigenspace of $\Delta_{S^{2n+1}}$ and $\mathcal{L}$, with eigenvalues respectively $\mu_{\ell,\ell'} := -(\ell+\ell')\,(\ell+\ell'+2n)$ and $\lambda_{\ell,\ell'} = -2\ell\ell' - n(\ell+\ell')$ ([Kl]).

The next task is proving that the joint spectral projection $P_{m,k}$ on h^n may be obtained as limit in the L^2-norm of an appropriate sequence of joint spectral projectors on S^{2n+1}.

Proposition 3.2. *Let f be a continuous function on h^n, with compact support. Take $m \in \mathbb{N}\setminus\{0\}$ and $k \in \mathbb{N}$. For every $\nu \in \mathbb{N}$ let $N(\nu) \in \mathbb{N}$ be such that*

$$\lim_{\nu\to+\infty} \frac{N(\nu)}{\nu} = m\,. \tag{3.6}$$

Then

$$||P_{m,k}f||_{L^2(h^n)} = \lim_{\nu\to+\infty} \frac{1}{\nu^{\frac{n}{2}}}||\pi_{k,N(\nu)-k}f_\nu||_{L^2(S^{2n+1})}\,,\ \textit{and} \tag{3.7}$$

$$||P_{-m,k}f||_{L^2(h^n)} = \lim_{\nu\to+\infty} \frac{1}{\nu^{\frac{n}{2}}}||\pi_{N(\nu)-k,k}f_\nu||_{L^2(S^{2n+1})}\,. \tag{3.8}$$

Proof. The scheme of the proof is similar to that of Proposition 4.4 in [Ca]. Since the higher dimensional case is more involved, we present the proof for more transparency.

Fix two integers $m > 0$ and $k \in \mathbb{N}$.

First of all, if $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, by writing $\mathbf{z} := \rho\underline{\eta}$ and $\mathbf{w} := \rho'\underline{\eta'}$, with $\rho, \rho' \in [0, +\infty)$ and $\underline{\eta}, \underline{\eta'} \in S^{2n-1}$, a simple computation yields

$$\begin{aligned}\Im m(\mathbf{z} \cdot \overline{\mathbf{w}}) = \rho\rho' \cdot \big(&\sin(\varphi_1 - \varphi_1') \sin\theta_{n-1} \sin\theta'_{n-1} \ldots\ldots \sin\theta_1 \sin\theta_1' \\ &+ \sin(\varphi_2 - \varphi_2') \sin\theta_{n-1} \sin\theta'_{n-1} \ldots\ldots \cos\theta_1 \cos\theta_1' + \ldots \\ &\ldots + \sin(\varphi_n - \varphi_n') \cos\theta_{n-1} \cos\theta'_{n-1}\big)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}|\mathbf{z} - \mathbf{w}|^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cdot \big(&\cos(\varphi_1 - \varphi_1') \sin\theta_{n-1} \sin\theta'_{n-1} \ldots \sin\theta_1 \sin\theta_1' \\ &+ \cos(\varphi_2 - \varphi_2') \sin\theta_{n-1} \sin\theta'_{n-1} \ldots \cos\theta_1 \cos\theta_1' + \ldots \\ &\ldots + \cos(\varphi_n - \varphi_n') \cos\theta_{n-1} \cos\theta'_{n-1}\big)\ .\end{aligned} \tag{3.10}$$

Now, by the symbol $\Phi^m_{k,k}$ we denote the joint eigenfunction for $\mathcal{L}$ and $i^{-1}\partial_t$ (for more details and an explicit expression see, for example, [FH, Chapitre V]). Orthogonality of joint spectral projectors yields

$$\begin{aligned}||P_{m,k}f||^2_{L^2(h^n)} &=< P_{m,k}f, f >_{L^2(h^n)}= \int_{h^n} f * \Phi^m_{k,k}(\mathbf{z}, t)\, \overline{f(\mathbf{z}, t)}\, d\mathbf{z}\, dt \\ &= \int_{h^n}\int_{h^n} \Phi^m_{k,k}\left(\mathbf{z} - \mathbf{w}, t - t' + \Im m(\mathbf{z} \cdot \overline{\mathbf{w}})\right) f(\mathbf{w}, t')\, d\mathbf{w}\, dt'\, \overline{f(\mathbf{z}, t)}\, d\mathbf{z}\, dt \\ &= m^n \int_{h^n}\int_{h^n} e^{i\, m(t-t'+\Im m(\mathbf{z}\cdot\overline{\mathbf{w}}))} L^{n-1}_k\left(m\, |\mathbf{z} - \mathbf{w}|^2\right) e^{-\frac{1}{2}m\, |\mathbf{z}-\mathbf{w}|^2} f(\mathbf{w}, t')\, d\mathbf{w}\, dt' \\ &\qquad \overline{f(\mathbf{z}, t)}\, d\mathbf{z}\, dt\, .\end{aligned}$$

Now we shall deal with the right-hand side in (3.7). For the sake of brevity we set

$$d\Phi_{(n)} := d\varphi_1\, , \ldots, d\varphi_n \quad \text{and}$$

$$d\Theta_{(n-1)} := \Pi_{j=1}^{n-1} \sin^{2j-1}\theta_j \cos\theta_j\, d\theta_j.$$

From the orthogonality of the joint spectral projectors $\pi_{\ell,\ell'}$ in $L^2(S^{2n+1})$ and from (3.5) we deduce

$$||\pi_{k,N(\nu)-k}f_\nu||^2_{L^2(S^{2n+1})} =< \pi_{k,N(\nu)-k}f_\nu, f_\nu >_{L^2(S^{2n+1})}$$

$$= \int_{S^{2n+1}} \left(\pi_{k,N(\nu)-k}f_\nu\right)(\Theta_{n-1}\,,\rho\,,\Phi_n,t)\,\overline{f_\nu(\Theta_{n-1}\,,\rho\,,\Phi_n,t)}\,d\sigma_{S^{2n+1}}$$

$$= \frac{n!}{2\pi^{n+1}\,\nu}\int_{A_\nu} \left(\pi_{k,N(\nu)-k}f_\nu\right)(\Theta_{n-1}\,,\frac{\rho}{\sqrt{\nu}}\,,\Phi_n,\frac{t}{\nu})\overline{\tilde{f}\,(\Theta_{n-1}\,,\rho\,,\Phi_n,t)}\left(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\right)^{2n-1}$$

$$\cos\frac{\rho}{\sqrt{\nu}}\,\rho^{2n-1}d\rho\,d\Theta_{(n-1)}\,d\Phi_{(n)}\,dt$$

$$= \frac{n!^2}{4\pi^{2n+2}\nu^2}\int_{A_\nu}\left(\int_{A_\nu} {}^b\mathbb{Z}_{k,N(\nu)-k}\left(< (\Theta_{n-1}\,,\frac{\rho}{\sqrt{\nu}},\Phi_n,\frac{t}{\nu})\,,(\Theta'_{n-1}\,,\frac{\rho'}{\sqrt{\nu}},\Phi'_n,\frac{t'}{\nu}) >\right)\right.$$

$$\left.\tilde{f}\left(\Theta'_{n-1}\,,\rho',\,\Phi'_n,t'\right)\left(\frac{\sin\frac{\rho'}{\sqrt{\nu}}}{\frac{\rho'}{\sqrt{\nu}}}\right)^{2n-1}\cos\frac{\rho'}{\sqrt{\nu}}\,\rho'^{2n-1}d\rho'\,d\Theta'_{(n-1)}\,d\Phi'_{(n)}\,dt'\right)$$

$$\overline{\tilde{f}\,(\Theta_{n-1}\,,\rho,\,\Phi_n,t)}\left(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\right)^{2n-1}\cos\frac{\rho}{\sqrt{\nu}}\rho^{2n-1}d\rho\,d\Theta'_{(n-1)}\,d\Phi_{(n)}\,dt$$

where the integration set A_ν is given by

$$A_\nu := \Big\{(\rho\,,\Theta_{n-1}\,,\Phi_n,t)\,:\,0\le\rho\le\frac{\pi}{2}\sqrt{\nu}\,,0\le\varphi_k\le 2\pi\,,k=1,\ldots,n\,,$$
$$0\le\theta_j\le\frac{\pi}{2}\,,j=1,\ldots,n-1\,,\,-\pi\nu\le t\le\pi\nu\Big\}\,. \tag{3.11}$$

Now by using (3.3) we compute the inner product in $\mathbb{C}^{n+1}$

$$< (\Theta_{n-1}\,,\frac{\rho}{\sqrt{\nu}},\Phi_{n-1},\frac{t}{\nu})\,,(\Theta'_{n-1}\,,\frac{\rho'}{\sqrt{\nu}},\Phi'_{n-1},\frac{t'}{\nu}) >=$$

$$= e^{i(\varphi_1-\varphi'_1)}\sin(\frac{\rho}{\sqrt{\nu}})\sin(\frac{\rho'}{\sqrt{\nu}})\sin\theta_{n-2}\sin\theta'_{n-2}\ldots\sin\theta_1\sin\theta'_1$$

$$+ e^{i(\varphi_2-\varphi'_2)}\sin(\frac{\rho}{\sqrt{\nu}})\sin(\frac{\rho'}{\sqrt{\nu}})\sin\theta_{n-2}\sin\theta'_{n-2}\ldots\cos\theta_1\cos\theta'_1$$

$$+\ldots+ e^{i(\varphi_{n-1}-\varphi'_{n-1})}\sin(\frac{\rho}{\sqrt{\nu}})\sin(\frac{\rho'}{\sqrt{\nu}})\cos\theta_{n-2}\cos\theta'_{n-2}$$

$$+ e^{i(t-t')\frac{1}{\nu}}\cos(\frac{\rho}{\sqrt{\nu}})\cos(\frac{\rho'}{\sqrt{\nu}})$$

$$= R_\nu e^{i\psi_\nu}\,,$$

where

$$R_\nu = 1 - \frac{1}{2\nu}\Bigg(\rho^2 + \rho'^2 - 2\rho\rho'\,\big(\cos(\varphi_1-\varphi_1')\sin\theta_{n-1}\sin\theta_{n-1}'\ldots\sin\theta_1\sin\theta_1' \\ + \cos(\varphi_2-\varphi_2')\sin\theta_{n-1}\sin\theta_{n-1}'\ldots\cos\theta_1\cos\theta_1' + \ldots \\ \ldots + \cos(\varphi_n-\varphi_n')\cos\theta_{n-1}\cos\theta_{n-1}'\big)\Bigg) + o(\frac{1}{\nu})\,,\ \nu\to+\infty\,,\ \text{and}$$

$$\psi_\nu = \arctan\Bigg(\frac{1}{\nu}\rho\rho'\,\big(\sin(\varphi_1-\varphi_1')\sin\theta_{n-1}\sin\theta_{n-1}'\ldots\sin\theta_1\sin\theta_1' \\ + \sin(\varphi_2-\varphi_2')\sin\theta_{n-1}\sin\theta_{n-1}'\ldots\cos\theta_1\cos\theta_1' + \ldots \\ \ldots + \sin(\varphi_n-\varphi_n')\cos\theta_{n-1}\cos\theta_{n-1}'\big) + \frac{t-t'}{\nu} + o(\frac{1}{\nu})\Bigg) \qquad \nu\to+\infty\,.$$

Thus as a consequence of (3.9) and (3.10) we have

$$R_\nu = \cos\left(\frac{1}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right) + o(\frac{1}{\nu}) \quad \text{and} \quad \psi_\nu = \frac{1}{\nu}(t-t') + \frac{1}{\nu}\Im m\,\mathbf{z}\,\overline{\mathbf{w}} + o(\frac{1}{\nu})\,,$$

so that formula (2.3) for the zonal function yields

$$ {}^b\mathbb{Z}_{k,N(\nu)-k}\left(<(\Theta_{n-1}\,,\frac{\rho}{\sqrt{\nu}},\Phi_n,\frac{t}{\nu})\,,(\Theta_{n-1}'\,,\frac{\rho'}{\sqrt{\nu}},\Phi_n',\frac{t'}{\nu})>\right) \\ = \frac{(N(\nu))^n}{\omega_{2n+1}}\,e^{i(N(\nu)-2k)\frac{1}{\nu}(t-t'+\Im m\,\mathbf{z}\,\bar{\mathbf{w}}+o(1))}\left(\cos\left(\frac{1}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right)\right)^{|N(\nu)-2k|} \\ P_k^{(n-1,|N(\nu)-2k|)}\left(\cos\left(\frac{2}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right)\right) + o(\frac{1}{\nu})\,,\ \nu\to+\infty\,.$$

By using condition (3.6) and the Mean Value Theorem, we easily check that

$$\frac{1}{\nu^n}||\pi_{k,N(\nu)-k}f_\nu||^2_{L^2(S^{2n+1})} = \mathcal{I}_\nu^M + \mathcal{I}_\nu^R\,,$$

where the remainder term $\mathcal{I}_\nu^R$ satisfies $\lim_{\nu\to+\infty}\mathcal{I}_\nu^R = 0$, while the main term $\mathcal{I}_\nu^M$ is given by

$$\mathcal{I}_\nu^M = \frac{n!^2}{4\omega_{2n+1}\pi^{2n+2}\nu^2}\int_{A_\nu}\left(\int_{A_\nu}\left(\frac{N(\nu)}{\nu}\right)^n e^{im(t-t'+\Im m\,\mathbf{z}\,\bar{\mathbf{w}})}\left(\cos\left(\frac{1}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right)\right)^{|N(\nu)-2k|}\right. \\ P_k^{(n-1,|N(\nu)-2k|)}\left(\cos\left(\frac{2}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right)\right)\tilde{f}\left(\rho',\,\Theta_{n-1}'\,,\Phi_n',t'\right)\left(\frac{\sin\frac{\rho'}{\sqrt{\nu}}}{\frac{\rho'}{\sqrt{\nu}}}\right)^{2n-1} \\ \left.\cos\frac{\rho'}{\sqrt{\nu}}\,\rho'^{2n-1}d\rho'\,d\Theta_{(n-1)}'\,d\Phi_{(n)}'\,dt'\right)\overline{\tilde{f}\left(\rho,\,\Theta_{n-1}\,,\Phi_n,t\right)}\left(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\right)^{2n-1} \\ \cos\frac{\rho}{\sqrt{\nu}}\,\rho^{2n-1}d\rho\,d\Theta_{(n-1)}\,d\Phi_{(n)}\,dt\,,\ \nu\to+\infty\,.$$

We shall now treat $\mathcal{I}_\nu^M$ by means of the Lebesgue Dominated Convergence Theorem. First of all, we extend the integration set in $\mathcal{I}_\nu^M$, (this may be done, since f has compact support and the integrand is periodic with respect to t), and we obtain

(3.12)
$$\begin{aligned}
\mathcal{I}_\nu^M =&\frac{n!^2}{4\pi^{2n+2}\omega_{2n+1}} \int_0^{+\infty}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{-\pi}^{\pi} \\
&\left(\int_0^{+\infty}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{-\pi}^{\pi}\left(\frac{N(\nu)}{\nu}\right)^n \quad e^{i\,m(t-t'-\Im m\mathbf{w}\,\bar{\mathbf{z}})}\right. \\
&\left(\cos\left(\frac{1}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right)\right)^{|N(\nu)-2k|} \quad P_k^{(n-1,|N(\nu)-2k|)}\left(\cos\left(\frac{2}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right)\right) \\
&\left. f\left(\rho',\,\Theta'_{n-1}\,,\Phi'_n,t'\right)\left(\frac{\sin\frac{\rho'}{\sqrt{\nu}}}{\frac{\rho'}{\sqrt{\nu}}}\right)^{2n-1} \cos\frac{\rho'}{\sqrt{\nu}}\,{\rho'}^{2n-1}d\rho'\,d\Theta'_{(n-1)}d\Phi'_{(n)}\,dt'\right) \\
&\overline{f\left(\rho,\,\Theta_{n-1}\,,\Phi_n,t\right)}\left(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\right)^{2n-1} \cos\frac{\rho}{\sqrt{\nu}}\,\rho^{2n-1}d\rho\,d\Theta_{(n-1)}\,d\Phi_{(n)}\,dt\,.
\end{aligned}$$

By using Lemma 2.1 and the Mehler-Heine formula as stated in Lemma 2.2 (with $N = N(\nu)+j-k$, $j-k = n-1$ and $x = \sqrt{\frac{N(\nu)-2k}{\nu}}|\mathbf{z}-\mathbf{w}|$), we may conclude as in Proposition 4.4 in [Ca].

The proof for (3.8) is completely analogous. $\square$

Theorem 3.3. *Let $n>2$. Take $m\in\mathbb{Z}\setminus\{0\}$ and $k\in\mathbb{N}$. Then*

$$(3.13)\quad ||P_{m,k}||_{(L^p(h^n),L^2(h^n))} \lesssim \begin{cases} C\ (2k+n)^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}\,|m|^{n\left(\frac{1}{p}-\frac{1}{2}\right)} & \text{if } 1\le p<\tilde{p} \\ C\ (2k+n)^{\frac{1}{4}-\frac{1}{2p}}\,|m|^{n\left(\frac{1}{p}-\frac{1}{2}\right)} & \text{if } \tilde{p}\le p\le 2, \end{cases}$$

where $\tilde{p}=2\frac{2n+1}{2n+3}$. Moreover, the estimates are sharp.

Proof. Take $m>0$ (the other case being analogous). For every $\nu\in\mathbb{N}$ let $N(\nu)\in\mathbb{N}$ be such that

$$\lim_{\nu\to+\infty}\frac{1}{\nu}\cdot N(\nu)=m\,.$$

Thus

$$\begin{aligned}
||P_{m,k}f||_{L^2(h^n)} &= \lim_{\nu\to+\infty}\frac{1}{\nu^{\frac{n}{2}}}||\pi_{k,N(\nu)-k}f_\nu||_{L^2(S^{2n+1})} \\
&\le \lim_{\nu\to+\infty}\left(\frac{N(\nu)}{\nu}\right)^{\frac{n}{2}}\left(\frac{2k\cdot(N(\nu)-k)}{N(\nu)}+n\right)^{\frac{n}{2}}||f_\nu||_{L^1(S^{2n+1})} \\
&= m^{\frac{n}{2}}(2k+n)^{\frac{n-1}{2}}\lim_{\nu\to+\infty}||f_\nu||_{L^1(S^{2n+1})} \\
&= m^{\frac{n}{2}}(2k+n)^{\frac{n-1}{2}}||f||_{L^1(h^n)}\,,
\end{aligned}$$

where we used first (3.7) and then Theorem 2.3 and Lemma 3.1.

In the same way, we see that

$$\begin{aligned}
||P_{m,k}f||_{L^2(h^n)} &= \lim_{\nu\to+\infty}\frac{1}{\nu^{\frac{n}{2}}}||\pi_{k,N(\nu)-k}f_\nu||_{L^2(S^{2n+1})}\\
&\le \lim_{\nu\to+\infty}\frac{1}{\nu^{\frac{n}{2}}}\left(\frac{2k\cdot(N(\nu)-k)}{N(\nu)}+n\right)^{-\frac{1}{2(2n+1)}}(N(\nu))^{\frac{n}{2n+1}}||f_\nu||_{L^{2\frac{2n+1}{2n+3}}(S^{2n+1})}\\
&\le (2k+n)^{-\frac{1}{2(2n+1)}}\lim_{\nu\to+\infty}\frac{1}{\nu^{\frac{n}{2}}}(N(\nu))^{\frac{n}{2n+1}}\nu^{\frac{n(2n-1)}{2(2n+1)}}||f||_{L^{2\frac{2n+1}{2n+3}}(h^n)}\\
&= (2k+n)^{-\frac{1}{2(2n+1)}}m^{\frac{n}{2n+1}}||f||_{L^{2\frac{2n+1}{2n+3}}(h^n)}\,.
\end{aligned}$$

An interpolation argument yields the thesis. Finally, sharpness follows from arguments in [KoR]. □

4. A restriction theorem on h^n

By applying the bounds proved in Section 2 we obtain a restriction theorem for the spectral projectors associated to the sub-Laplacian L on h^n. Our theorem improves in some cases a previous result due to Thangavelu ([Th1]). More precisely, let Q_N be the spectral projection corresponding to the eigenvalue N associated to L on h^n, that is

$$Q_N f := \sum_{(2k+n)|m|=N} P_{m,k}f\,,$$

where $P_{m,k}$ is the joint spectral projection operator introduced in the previous section. We look for estimates of the type

$$||Q_N||_{(L^p(h^n),L^2(h^n))}\le C\,N^{\sigma(p,n)}\,, \tag{4.1}$$

for all $1\le p\le 2$, where the exponent σ is in general a convex function of $\frac{1}{p}$.
In [Th91] Thangavelu proved that

$$||Q_N||_{(L^p(h^n),L^2(h^n))}\le C\,N^{n(\frac{1}{p}-\frac{1}{2})}d(N)^{\frac{1}{p}-\frac{1}{2}}\,,\qquad 1\le p\le 2\,, \tag{4.2}$$

where $d(N)$ is the divisor-type function defined by

$$d(N) := \sum_{2k+n|N}\frac{1}{2k+n}\,, \tag{4.3}$$

and the estimate is sharp for $p=1$. By $a|b$ we mean that a divides b.

Thangavelu also proved that when $N=nR$, with $R\in\mathbb{N}$, then

$$C\,N^{n(\frac{1}{p}-\frac{1}{2})}\le ||Q_N||_{(L^p(h^n),L^2(h^n))}\,,\qquad 1\le p\le 2\,.$$

Here we show that there exist arithmetic progressions a_N in $\mathbb{N}$ such that the estimate for $||Q_{a_N}||_{(p,2)}$ is sharp and better than (4.2) for $1<p<2$.

Proposition 4.1. *Let $n\ge 1$. Let N be any positive integer number. Then for every $1\le p\le 2$*

$$||Q_N||_{(L^p(h^n),L^2(h^n))}\le C\,N^{n(\frac{1}{p}-\frac{1}{2})}d(N)^{\rho(\frac{1}{p},n)}\,, \tag{4.4}$$

where ρ is defined by

$$\rho(\frac{1}{p},n) := \begin{cases} \frac{1}{2} & \text{if } 1 \leq p < \tilde{p} \\ (2n+1)\left(\frac{1}{2p}-\frac{1}{4}\right) & \text{if } \tilde{p} \leq p \leq 2, \end{cases} \tag{4.5}$$

with $\tilde{p} = 2\frac{2n+1}{2n+3}$, and $d(N)$ is given by (4.3).

Proof. For $p = 1$ our estimate coincide with (4.2); nonetheless we give a different, simpler proof:

$$\begin{aligned} ||Q_N f||^2_{L^2(h^n)} &= || \sum_{(2k+n)|m|=N} P_{m,k} f||^2_{L^2(h^n)} = \sum_{(2k+n)|m|=N} ||P_{m,k} f||^2_{L^2(h^n)} \\ &\leq C \sum_{(2k+n)|m|=N} m^n (2k+n)^{n-1} ||f||^2_{L^1(h^n)}\,, \\ &\leq C N^n \sum_{2k+n|N} \frac{1}{2k+n} ||f||^2_{L^1(h^n)}\,, \end{aligned}$$

whence

$$||Q_N||_{(L^1,L^2)} \leq C N^{\frac{n}{2}} \left(d(N)\right)^{\frac{1}{2}}\ . \tag{4.6}$$

For $p = 2$ the bound is obvious, since Q_N is an orthogonal projector. Finally, for $p = \tilde{p}$ one has

$$\begin{aligned} ||Q_N f||^2_{L^2(h^n)} &= \sum_{(2k+n)|m|=N} ||P_{m,k} f||^2_{L^2(h^n)} \\ &\leq C \sum_{(2k+n)|m|=N} (2k+n)^{-\frac{1}{2n+1}} |m|^{\frac{2n}{2n+1}} ||f||^2_{L^{\tilde{p}}(h^n)}\,, \\ &= C N^{\frac{2n}{2n+1}} \sum_{2k+n|N} (2k+n)^{-1} ||f||^2_{L^{\tilde{p}}(h^n)}\,, \end{aligned}$$

whence

$$||Q_N||_{(L^{\tilde{p}},L^2)} \leq C N^{\frac{n}{2n+1}} \left(d(N)\right)^{\frac{1}{2}}\ . \tag{4.7}$$

Thus by applying the Riesz-Thorin interpolation theorem to (4.6) and to (4.7) we get (4.4). □

Remark 4.2. Observe that estimate (4.4) is better than (4.2) only when $d(N) < 1$.

Thus, on the one hand we are led to seek arithmetic progressions $\{N_m\}$ on which the divisor function $d(N_m)$, whose behaviour is in general highly irregular, is strictly smaller than one. On the other one, we are led to inquire about the average size of the norm of Q_N.
We remark that, if $n = 1$ then $d(N)$ is necessarily greater than one.

Remark 4.3. Proposition 4.1 reveals the existence of a critical point $\tilde{p} \in (1, 2)$, where the form of the exponent of the eigenvalue N in (4.1) changes.

In the following we list some cases in which estimate (4.4) really improves the result in [Th1]. First of all, when $n \geq 2$ and N is a prime number, Proposition 4.1 yields the following sharp result.

Proposition 4.4. *Let $n > 2$, n odd. Let N be a prime number. Then for every $1 \leq p \leq 2$*

$$||Q_N||_{(L^p(h^n),L^2(h^n))} \leq \begin{cases} C\, N^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}\,, & \text{if } 1 \leq p < \tilde{p} \\ C\, N^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})} & \text{if } \tilde{p} \leq p \leq 2, \end{cases} \tag{4.8}$$

with $\tilde{p} = 2\frac{2n+1}{2n+3}$. Moreover, the above estimate is sharp.

Proof. (4.8) follows directly from (4.4).

Furthermore, since in this case

$$||Q_N||_{(L^p(h^n),L^2(h^n))} \sim ||P_{1,\frac{N-n}{2}}||_{(L^p(h^n),L^2(h^n))}\,, \quad 1 \leq p \leq 2\,,$$

sharpness follows from Theorem 3.3. □

Proposition 4.4 may be generalized to the case $N = r^{k_0}$, where $k_0 \in \mathbb{N}$ and r varies in the set of all prime numbers.

Proposition 4.5. *Let $n \geq 2$ be odd. Fix a positive integer number k_0. Set $N_r = r^{k_0}$, where r varies in the set of all prime numbers. Then for every $1 \leq p \leq 2$*

$$||Q_{N_r}||_{(L^p(h^n),L^2(h^n))} \leq \begin{cases} C\, {N_r}^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2k_0}}\,, & \text{if } 1 \leq p < \tilde{p} \\ C\, {N_r}^{(n-\frac{1}{2k_0}(2n+1))(\frac{1}{p}-\frac{1}{2})} & \text{if } \tilde{p} \leq p \leq 2, \end{cases} \tag{4.9}$$

with $\tilde{p} = 2\frac{2n+1}{2n+3}$. Moreover, (4.9) is sharp.

Proof. (4.9) follows directly from (4.4), since

$$d(N_r) = \frac{1}{r} + \frac{1}{r^2} + \ldots + \frac{1}{r^{k_0}} \leq \frac{2}{r}.$$

To prove that (4.9) is sharp, take the joint eigenfunction f_0 for L and $i^{-1}\partial_t$, with eigenvalues, respectively, $(2k+n)m = N_r$ and $m = r^{k_0-1}$, yielding the sharpness for the joint spectral projection $P_{r^{k_0-1},\frac{r-n}{2}}$, that is such that

$$||P_{r^{k_0-1},\frac{q-n}{2}}||_{(p,2)} \sim \frac{||f_0||_{p'}}{||f_0||_2}\,.$$

Now we have

$$\begin{aligned} ||Q_N||_{\left(L^2(h^n),L^{p'}(h^n)\right)} &\geq \frac{||Q_N f_0||_{L^{p'}}}{||f_0||_{L^2}} = \frac{||f_0||_{L^{p'}}}{||f_0||_{L^2}} \sim ||P_{r^{k_0-1},\frac{r-n}{2}}||_{(p,2)} \\ &\sim Cr^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}\left(r^{k_0-1}\right)^{n(\frac{1}{p}-\frac{1}{2})} \sim Cr^{-\frac{1}{2}}r^{k_0 n(\frac{1}{p}-\frac{1}{2})} \\ &\sim C{N_r}^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2k_0}} \end{aligned}$$

for all $1 \leq p \leq \tilde{p}$. For $\tilde{p} \leq p \leq 2$ an analogous estimate hold, so that (4.9) is sharp. □

We shall now consider integers of the form $N_\ell := {q_0}^\ell$, where q_0 is a fixed prime number and $\ell \in \mathbb{N}$. The argument of the previous proposition also proves the following.

Proposition 4.6. *Let $n = 2$ or $n > 2$ odd. For $n = 2$ let $q_0 = 2$, for $n > 2$ let q_0 be a prime number strictly greater than 2. Set $N_\ell := {q_0}^\ell$, $\ell \in \mathbb{N}$.*
Then

$$||Q_{N_\ell}||_{(L^p(h^n),L^2(h^n))} \le C\, {N_\ell}^{n(\frac{1}{p}-\frac{1}{2})} \quad \text{if } 1 \le p \le 2. \tag{4.10}$$

Moreover, (4.10) is sharp.

The above examples show the highly irregular behaviour of $d(N)$, and therefore of $||Q_N||_{p,2}$. In order to smooth out fluctuations we introduce appropriate averages of joint spectral projectors. More precisely, we define for $N \in \mathbb{N}$

$$\Pi_N f := \sum_{L=n}^{N} \sum_{(2k+n)|m|=L} P_{m,k} f \tag{4.11}$$

and ask what is the behaviour of $||M_N||_{(p,2)}$, where

$$M_N f := \frac{1}{N} \Pi_N f\, . \tag{4.12}$$

For $p = 1$ Theorem 3.3 and orthogonality yield

$$\begin{aligned}
||\Pi_N f||^2_{L^2(h^n)} &= ||\sum_{L=n}^{N} \sum_{(2k+n)|m|=L} P_{m,k} f||^2_{L^2(h^n)} \\
&= \sum_{(k,m):\,(2k+n)|m|\le N} ||P_{m,k} f||^2_{L^2(h^n)} \\
&\le C \sum_{(k,m):\,(2k+n)|m|\le N} (2k+n)^{n-1}\, |m|^n ||f||^2_{L^1(h^n)} \\
&\le C \sum_{m=1}^{N} m^n \sum_{2k+n=n}^{\left[\frac{N}{m}\right]} (2k+n)^{n-1} ||f||^2_{L^1(h^n)} \le C\, N^n \cdot N ||f||^2_{L^1(h^n)}\, ,
\end{aligned}$$

whence

$$||\Pi_N||_{(1,2)} \le N^{\frac{n+1}{2}}\, . \tag{4.13}$$

The trivial $L^2 - L^2$ estimate and Riesz-Thorin interpolation yield

$$||\Pi_N||_{(p,2)} \le C\, N^{(n+1)(\frac{1}{p}-\frac{1}{2})} \qquad 1 \le p \le 2 \tag{4.14}$$

Observe that by using Theorem 3.3 we may obtain the following estimate in the critical point $\tilde{p}$

$$\begin{aligned}
||\Pi_N f||^2_{L^2(h^n)} &= \sum_{(k,m):\,(2k+n)|m|\le N} ||P_{m,k} f||^2_{L^2(h^n)} \\
&\le C \sum_{(k,m):\,(2k+n)|m|\le N} (2k+n)^{2\alpha}\, m^{2\beta} ||f||^2_{L^{\tilde p}(h^n)} \\
&= C \sum_{m=1}^{N} m^{2\beta} \sum_{2k+n=n}^{\frac{N}{m}} (2k+n)^{2\alpha} ||f||^2_{L^{\tilde p}(h^n)} = N^{2\alpha+1} \sum_{m=1}^{N} m^{2\beta-2\alpha-1} ||f||^2_{L^{\tilde p}(h^n)} \\
&\le C\, N^{2\alpha+2} ||f||^2_{L^{\tilde p}(h^n)}\,,
\end{aligned}$$

where we used the fact that $2\beta - 2\alpha = 1$ for all $1 \le p \le \tilde p$, with $\alpha = \alpha(\frac{1}{p}, n)$ and $\beta = \beta(\frac{1}{p}, n)$ given by (2.6) and (2.7).

Thus

$$||\Pi_N||_{(\tilde p,2)} \le C\, N^{\alpha+1} = C\, N^{\frac{2n+\frac{1}{2}}{2n+1}}\,. \tag{4.15}$$

A comparison between (4.14) and (4.15) shows that in the critical point the estimate given by Riesz-Thorin interpolation is better than the bound obtained by summing up the estimates for joint spectral projections.

Thus we obtain the following result.

Proposition 4.7. *Let $n \ge 1$. The following $L^p - L^2$ bounds hold for Π_N and for the average projection operators M_N*

$$||\Pi_N||_{(L^p(h^n),L^2(h^n))} \le C\, N^{(n+1)(\frac{1}{p}-\frac{1}{2})} \quad \text{if } 1 \le p \le 2.$$

and

$$||M_N||_{(L^p(h^n),L^2(h^n))} \le C\, N^{(n+1)(\frac{1}{p}-\frac{1}{2})-1} \quad \text{if } 1 \le p \le 2.$$

A similar proof also yields the following result about the operators E_{N_1,N_2}, where

$$E_{N_1,N_2} := \Pi_{N_2} - \Pi_{N_1}\,, \qquad N_1\,, N_2 \in \mathbb{N}\,,\; N_2 > N_1\,.$$

Proposition 4.8. *Let $n \ge 1$. Then*

$$||E_{N_1,N_2}||_{(L^p(h^n),L^2(h^n))} \le C\; (N_2^n (N_2 - N_1))^{(\frac{1}{p}-\frac{1}{2})} \quad \text{for all } 1 \le p \le 2.$$

Remark 4.9. This should be compared to Proposition 3.8 in [M].

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